INCREASING AND DECREASING FUNCTIONS

A function f (x) is said to be strictly **increasing** function on (a, b) if

$x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ ***for all*** $x_1, x_2 \in (a, b)$

A function f (x) is said to be a strictly **decreasing** function on (a, b) if

$x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$ ***for all*** $x_1, x_2 \in (a, b)$

Thus, f (x) is strictly decreasing on (a, b) if the values of f (x) decrease with the increase in the values of x.

Remember:

(i) If $f'(x) \geq 0$ for all x in D (a subset of R), then, $f(x)$ is increasing in D.

(ii) If $f'(x) \leq 0$ for all x in D, then, $f(x)$ is decreasing in D.

(iii) If $f'(x) \geq 0$ for all x in some open interval (a, b), then, $f(x)$ is increasing $[a, b] \cap D_f$

(iv) If $f'(x) \leq 0$ for all x in some open interval (a, b), then, $f(x)$ is decreasing in $[a, b] \cap D_f$

EXAMPLES

1. Show that f (x) = log (sin x) is increasing on $(0, \pi/2)$ and decreasing on $(\pi/2, \pi)$

Sol. $f(x) = \log \sin x \Rightarrow f'(x) = \cot x$.

Now, $0 < x < \pi/2 \Rightarrow \cot x > 0 \Rightarrow f'(x) > 0$.

And, $\pi/2 < x < \pi \Rightarrow \cot x < 0 \Rightarrow f'(x) < 0$.

2. Prove that the function $f(x) = x^3 - 3x^2 + 3x - 100$ is increasing on R

Sol. We have $f(x) = x^3 - 3x^2 + 3x - 100$

$\therefore f'(x) = 3x^2 - 6x + 3 = 3(x-1)^2$. Now, $x \in R$

$\Rightarrow (x-1)^2 \geq 0 \Rightarrow f'(x) \geq 0$. Thus, $f'(x) \geq 0$ for all $x \in R$.

Hence, f (x) is increasing on R.

3. Which of the following functions are decreasing on $(0, \pi/2)$?

(a) cos x (b) cos 2x (c) tan x (d) cos 3x

Sol. (a) We have $f(x) = \cos x \therefore f'(x) = -\sin x$.

Now, $x \in (0, \pi/2) \Rightarrow \sin x > 0 \Rightarrow -\sin x < 0$

$\Rightarrow f'(x) < 0$. So, f (x) is decreasing on $(0, \pi/2)$.

(b) Let $f(x) = \cos 2x$. Then $f'(x) = -2 \sin 2x$.

Now, $x \in (0, \pi/2) \Rightarrow 0 < x < \pi/2 \Rightarrow 0 < 2x < \pi \Rightarrow \sin 2x > 0$

$\Rightarrow -2 \sin 2x < 0 \Rightarrow f'(x) < 0$.

So, f (x) is decreasing on $(0, \pi/2)$.

(c) Let f (x) = tan x. Then $f'(x) = \sec^2 x$.

Now, $x \in (0, \pi/2) \Rightarrow \sec^2 x > 0 \Rightarrow f'(x) > 0$

So, f(x) is increasing on $(0, \pi/2)$

(d) Let f (x) = cos 3x. Then $f'(x) = -3 \sin 3x$.

Now, $x \in (0, \pi/2) \Rightarrow 0 < x < \pi/2 \Rightarrow 0 < 3x < 3\pi/2$

$\Rightarrow$ sin 3x can be positive as well as negative

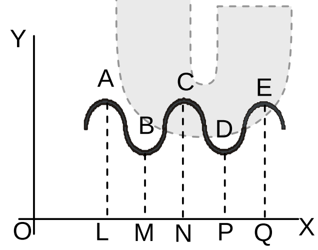
$\Rightarrow f(x) = -3 \sin 3x$ can be positive as well as negative.

So, f (x) is neither increasing nor decreasing on $(0, \pi/2)$

MAXIMA AND MINIMA

Maximum Value: A continuous function f (x) is said to have a maximum value for x = a, if f (a) is greater than any other value of f (x) lying in small neighbourhood of x = a.

Minimum Value: A continuous function f (x) is said to have a minimum value of x = a, if f (a) is smallest of all f (x) lying in small neighbourhood of x = a.



The following points shall be very useful

- If the sum of a few quantities is given, their product is maximum when they are equal.
- If the product of a few quantities is given, their sum is minimum when they are equal.
- The arithmetic mean of any number of quantities is greater than or equal to their geometric mean. i.e. AM $\geq$ GM always
- The point on a curve closest to a given line will be the one at which the tangent is parallel to the line given.

Extreme Value: Either a maximum value or a minimum value f (a) of the function f (x) is said to be extreme value.

Note: The tangent at maximum or minimum point of the curve is parallel to x-axis.

Stationary Value: If f '(a) = 0, then f (a) is said to be stationary value which need not be an extreme value.

Note: Every extreme value is stationary but every stationary value need not be an extreme value.

Example: Let $f(x) = x^5 - 5x^4 + 5x^3 - 1 \Rightarrow f'(x) = 5x^4 - 20x^3 + 15x^2 \Rightarrow f'(0) = 0$.

$\therefore$ f (0) is a stationary value but f (0) is not an extreme value because $f''(0) = 0$, $f'''(0) \neq 0$.

Greatest Value: The greatest value of a function in an interval (a, b) is either a maximum value of f (x) at a point inside the interval or end value (i.e., at x = a or x = b) of f (x) which ever is greater.

Least Value: The least value of f (x) in an interval (a, b) is either a minimum value of f (x) at a point inside the interval or an end value (i.e., at x = a or x = b) of f (x) which ever is smaller.

ALGORITHM FOR DETERMINING EXTREME VALUES OF A FUNCTION

From the above test criteria we obtain the following rule for determining maxima and minima of f (x)

Step I. Find f ' (x)

Step II. Put f ' (x) = 0 and solve this equation for x. Let $c_1, c_2 \ldots c_n$ be the roots of this equation. $c_1, c_2 \ldots c_n$ are stationary values of x and these are the possible points where the function can attain a local maximum or a local minimum So we test the function at each one of these points.

Step III. Find $f''(c_1)$

- If $f''(c_1) < 0$, then $x = c_1$ is a point of local maximum.
- If $f''(c_1) > 0$, then $x = c_1$ is a point of local minimum
- If $f''(c_1) = 0$, we must find $f'''(x)$ and substitute in it c_1 for x.
- If $f'''(c_1) \neq 0$, then $x = c_1$ is neither a point of local maximum nor a point of local minimum and is called the point of inflection.
- If $f'''(c_1) = 0$, we must find $f^4(x)$ and substitute in it c_1 for x.
- If $f^4(c_1) < 0$, then $x = c_1$ is a point of local maximum and if $f^4(c_1) > 0$, then c_1 is a point of local minimum.
- If $f^4(c_1) = 0$, we must find $f^5(x)$ and so on.
- Similarly the values of $c_2, c_3 \ldots$.. may be tested.

Point of inflection: A point of inflection is a point at which a curve is changing concave upward to concave downward or vice-versa.

A curve y = f (x) has one of its points x = c as an inflection point, If $f''(c) = 0$ or is not defined and if $f''(x)$ changes sign as x increases through x = c.

The later condition may be replaced by $f'''(c) \neq 0$ when $f'''(c)$ exists.

Thus x = c is a point of inflection if $f''(c) = 0$ and $f'''(c) \neq 0$.

Properties of Maxima and Minima

(i) If f (x) is continuous function in its domain, then at least one maximum and one minimum must lie between two equal values of x.

(ii). Maxima and minima occur alternately, that is, between two maxima there is one minimum and vice-versa.

EXAMPLES

1. Find the maximum and the minimum values of $f(x) = x + \sin 2x$ in the interval $[0, 2\pi]$.

Sol. We have $f(x) = x + \sin 2x$. So, $f'(x) = 1 + 2\cos 2x$.

For stationary points, we have

$f'(x) = 0 \Rightarrow 1 + 2\cos 2x = 0 \Rightarrow \cos 2x = -1/2$

$\Rightarrow 2x = 2\pi/3$, or $2x = 4\pi/3$ (as $0 \le x \le 2\pi \therefore 0 \le 2x \le 4\pi$)

$\Rightarrow x = \pi/3$ or $x = 2\pi/3$.

Now, $f(0) = 0 + \sin 0 = 0$,

$f(\pi/3) = \pi/3 + \sin 2\pi/3 = \pi/3 + \sqrt{3}/2$,

$f(2\pi/3) = 2\pi/3 + \sin 4\pi/3 = 2\pi/3 - \sqrt{3}/2$

and $f(2\pi) = 2\pi + \sin 4\pi = 2\pi + 0 = 2\pi$.

Of these values, the maximum value is 2π and the minimum value is 0.

Thus the maximum value of f (x) is 2π and the minimum value is 0.

2. Find the maximum profit that a company can make, if the profit function is given by $P(x) = 41 + 24x - 18x^2$.

Sol. $P(x) = 41 + 24x - 18x^2$.

$\Rightarrow \dfrac{dP(x)}{dx} = 24 - 36x$ and $\dfrac{d^2P(x)}{dx^2} = -36$.

For maximum or minimum, $\dfrac{dP(x)}{dx} = 0$

$\Rightarrow 36 - 36x = 0 \Rightarrow x = 2/3$.

Now, $\left(\dfrac{d^2P(x)}{dx^2}\right)_{x=2/3} = -36 < 0$.

Profit is maximum when $x = 2/3$.

Maximum profit = (value of P (x) at x = 2/3) $= 41 + 24 \times (2/3) - 18(2/3)^2 = 49$.

3. Show that the maximum value of $(1/x)^x$ is $e^{1/e}$.

Sol. Let $y = (1/x)^x = x^{-x}$. Then $\log y = -x \log x$

$$\therefore \frac{1}{y}\frac{dy}{dx} = -(1 + \log x) \text{ or } \frac{dy}{dx} = -y(1 + \log x)$$

$$\text{And, } \frac{d^2y}{dx^2} = -\frac{dy}{dx}(1 + \log x) - y/x = y(1 + \log x)^2 - \frac{y}{x}\frac{d^2y}{dx^2}$$

$$= x^{-x}(1 + \log x)^2 - x^{-x}/x. = x^{-x}(1 + \log x)^2 - x^{-x-1}.$$

For maximum and minimum, $\frac{dy}{dx} = 0$

$$\therefore \frac{dy}{dx} = 0 \Rightarrow -y(1 + \log x) = 0 \Rightarrow 1 + \log x = 0 \Rightarrow \log x = -1$$

$$\Rightarrow x = e^{-1} = 1/e \quad [\text{as } \log_e A = b \Rightarrow A = e^b]$$

$$\text{Also, } \left(\frac{d^2y}{dx^2}\right)_{x=1/e} = -\left(\frac{1}{e}\right)^{-1/e}\left(1+\log\frac{1}{e}\right)^2 - \left(\frac{1}{e}\right)^{-1/e-1}$$

$$= -(e^{-1})^{-1/e}(1 - \log e)^2 - (e^{-1})^{-1/e-1}$$

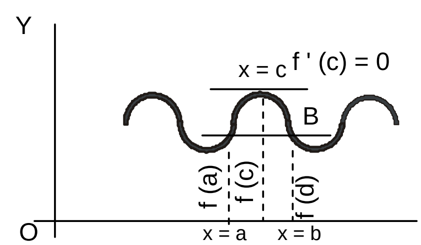
$$= -e^{1/e}(1 - 1)^2 - e^{1/e+1} = -e^{1/e+1} < 0$$

So, $x = 1/e$ is a point of local maximum.

The local maximum value of y is given by $y = (e)^{1/e}$

MEAN VALUE THEOREMS

ROLLE'S THEOREM



Let f (x) be a function such that

i. f (x) is continuous in [a, b]

ii. f ' (x) exists for every point in (a, b)

iii. f (a) = f (b)

Then, there exists at least one point $c \in (a, b)$ such that

$$f'(c) = 0$$

Interpretation of Rolle's theorem

Geometric: Let f (x) be a function defined on [a, b] such that the curve y = f (x) is continuous between points (a, f (a)) and (b, f (b)); at every point on the curve, except at the end points, it is possible to draw a unique tagnent and ordinates at x = a and x = b are equal. Then there exists at least one point on the curve where tangent is parallel to x-axis.

Algebraic: Let f (x) be a polynomial with a and b as its two zeros. Then f (a) = f (b) = 0. Also a polynomial function is everywhere continuous and differentiable. Therefore by Rolle's Theorem there exists at least one point $c \in (a, b)$ such that f ' (c) = 0 → x = c is root of f ' (x) = 0 or x = c is a zero of f ' (x).

Hence, between two zeros of a polynomial f (x) there exists at least one zero of f ' (x)

EXAMPLES

1. Let $f(x) = x(x + 3)e^{-x/2}$, then how many values of x exist in (– 3, 0) such that f ' (x) = 0?

(a) no (b) one (c) two (d) three

Sol. The given function is $f(x) = x(x + 3)e^{-x/2}$. f (x) is continuous in [– 3, 0].

$f'(x) = (2x + 3)e^{-x/2} - \frac{1}{2}e^{-x/2}(x^2 + 3x) = \frac{1}{2}(6 + x - x^2)e^{-x/2}$

$\therefore$ f ' (x) exists in (–3, 0).

$\therefore f(-3) = (-3)(-3 + 3)e^{3/2} = 0$. $f(0) = 0(0 + 3)e^0 = 0$.

$\therefore$ f (– 3) = f (0)

$\therefore$ Rolle's Theorem is applicable. At least one $x \in (-3, 0)$ such that f ' (x) = 0

$\therefore 1/2\, e^{-x/2}(6 + x - x^2) = 0 \rightarrow x = 3, -2$. Only $-2 \in (-3, 0)$.

$\therefore$ The correct answer is (b)

FIRST MEAN VALUE THEOREM (LAGRANGE'S MEAN VALUE THEOREM)

If f (x) is a function such that

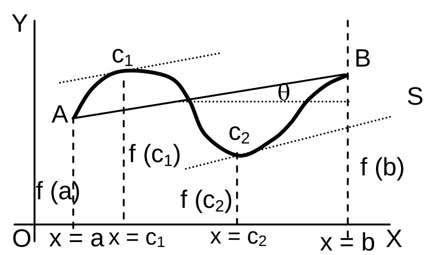
i. f (x) is continuous in [a, b]

ii. ii. f ' (x) exists in (a, b)

Then there exists at least one point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}, \text{ or}$$

$$f(b) = f(a) + (b - a) f'(c)$$



Another Form – $f(b) = f(a) + (b - a) f'\{a + (b - a)\theta\}, 0 < \theta < 1$

Let b – a = h, then, $f(a + h) = f(a) + h f'(a + h\theta), 0 < \theta < 1$

$\frac{f(b) - f(a)}{b - a}$ = Slope of AB = Slope of S_1 or S_2 = $f'(c_1)$ or, $f'(c_2)$

Geometrical Interpretation of Lagrange's theorem

If interpreted geometrically, this theorem means that there exists a point (c, f (c)), on the curve y = f (x) at which the tangent to curve is parallel to the chord joining (a, f (a)) and (b, f (b)).

TANGENTS AND NORMALS

A. Rule to find the equation of the tangent to the curve y = f(x) at the given point P(x_1, y_1).

(i) Find $\frac{dy}{dx}$ from the given equation $y = f(x)$

(ii) Find the value of $\frac{dy}{dx}$ at the given point p(x_1, y_1), let $m = \left(\frac{dy}{dx}\right)_{at(x_1, y_1)}$

(iii) The equation of the required tangent is $y - y_1 = m(x - x_1)$.

Remark: If $\left(\frac{dy}{dx}\right)_{at(x_1, y_1)} = 0$ then, the tangent is parallel to Y-axis and its equation is $x = x_1$

B. Rule to find the equation of the normal to the curve y = f(x) at the given point P(x_1, y_1)

(i) Find $\frac{dy}{dx}$ from the given equation $y\ f(x)$.

(ii) Find the value of $\left(\frac{dy}{dx}\right)$ at the given point P(x_1, y_1).

(iii) If m is the slope of the normal at the point P, then $\frac{1}{\left(\frac{dy}{dx}\right)_{\substack{x=x_1 \\ y=y_1}}}$

(iv) The equation of required normal is $y - y_1 = m\ (x - x_1)$

Remark: If $\left(\frac{dy}{dx}\right)_{at(x_1, y_1)} = 0$, then the equation of the normal at P is $x = x_1$ and

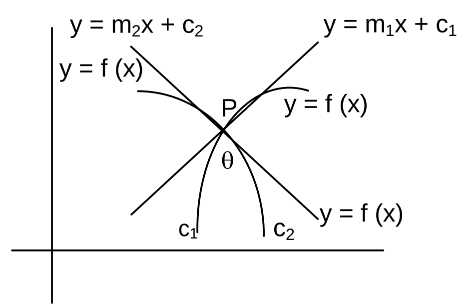
if $\left(\frac{dy}{dx}\right)_{at(x_1, y_1)} = \infty$, then the equation of the normal at P is $y = y_1$

<u>Angle of intersection of two curves:</u> *By the angle of intersection of two curves, we mean angle between the tangents at their common point of intersection.*

Let P be any point of intersection of two curves y = f (x) and y = g (x) and the equation of tangents at P are $y = m_1 x + C_1$ and $y = m_2 x + C_2$.

Then angle between these lines is

$$\tan\theta = \pm \frac{m_1 - m_2}{1 + m_1 m_2}$$



The positive value of tan θ would give the acute angle whereas, the negative value of tan θ would give the obtuse angle between the curves.

Note:

1. If the curves touch each other, then $m_1 = m_2$, $\theta = 0 \Rightarrow \tan\theta = 0$.

2. If the curves cut orthogonally, then $m_1 m_2 = -1$, $\theta = 90^o \Rightarrow \tan\theta = \pi/2$

EXAMPLES

1. Show that the condition that the curves $ax^2 + by^2 = 1$ … (i) and $a'x^2 + b'y^2 = 1$… (ii) should intersect orthogonally is that $1/a - 1/b = 1/a' - 1/b'$.

Sol. Let (x_1, y_1) be the point of intersection of the curves.

Then $ax_1^2 + by_1^2 = 1$… (iii) and $a'x_1^2 + b'y_1^2 + b'y_1^2 = 1$… (iv).

Differentiating (i) w.r.t. x, we get,

$$2ax + 2by\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{ax}{by} \Rightarrow m_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = -\frac{ax_1}{by_1} \text{ … (v).}$$

Differentiating (ii) w.r.t. x, we get

$$2a'x + 2b'y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{a'x}{b'y} \Rightarrow m_2 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = -\frac{a'x_1}{b'y_1} \text{ … (vi).}$$

The two curves will intersect orthogonally, if $m_1 m_2 = -1$

$$\Rightarrow -\frac{ax_1}{by_1} x - \frac{a'x_1}{b'y_1} = -1 \Rightarrow aa'x_1^2 = -bb'y_1^2 \text{ … (vii).}$$

Subtracting (iv) from (iii), we get,

$(a - a')x_1^2 = -(b - b')y_1^2$ … (vii).

Dividing (viii) by (vii), we get,

$(a - a') / aa' = (b - b')/bb'$

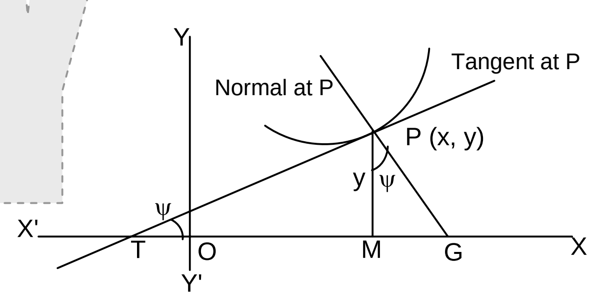
$\Rightarrow 1/a - 1/b = 1/a' - 1/b'$

Length of Cartesian Tangent, Normal, Sub-tangent and Sub-normal

Let (x, y) be any point P on the curve y = f (x)

Tangent = PT = MP cosec ψ = MP $\sqrt{1+\cos^2\Psi}$

Tangent, PT = $y\sqrt{1+\left(\frac{dx}{dy}\right)^2}$



Sub tangent TM = PM cot ψ = $y\frac{dx}{dy}$.

Sub normal = MG = PM tan ψ = $y\frac{dy}{dx}$

LENGTH OF	VALUE
Tangent	$\frac{y}{y'}\sqrt{1+(y')^2}$
Normal	$y\sqrt{1+(y')^2}$
sub tangent	$\frac{y}{y'}$
sub normal	$y\,y'$

ASYMPTOTES

A Straight line, at a finite distance from origin, is said to be an asymptote of the curve y = f (x) if the perpendicular distance of the point P on the curve from the line tends to zero when x or y or both tends to infinity.

Working Rule

To find the asymptotes of the curve which is -

- **parallel to x-axis -**

  Equate the coefficient of highest power of the x to zero.

  If this coefficient is constant, then there is no asymptotes parallel to x-axis (horizontal).

- **parallel to y-axis –**

  Equate the coefficient of highest power of y to zero.

  If this coefficient is constant, then there is no asymptotes parallel to y-axis (vertical).

EXAMPLES

1. For the curve $y = \frac{2x^2-5x+8}{5x^2+3x-2}$ which of the following is false?

(a) y = 2/5 is a horizontal asymptote (b) x = 2/5 is a vertical asymptote

(c) x = 1 is a vertical asymptote (d) x = – 1 is a vertical asymptote

Sol. The given curve is $(5x^2 + 3x - 2) = 2x^2 - 5x + 8$ or, $x^2 (5y - 2) + \ldots = 0$

Equating to zero the coefficient of x^2, we get

$5y - 2 = 0 \Rightarrow y = 2/5$

$\therefore$ y = 2/5 is a horizontal asymptote.

Now from the given equation $y (5x - 2) (x + 1) - (2x^2 - 5x + 8) = 0$

Equating to zero the coefficient of y, we get,

$5x - 2 = 0,\ x + 1 = 0 \rightarrow x = 2/5,\ x = -1$

$\therefore$ Vertical asymptotes are x = 2/5, x = – 1.

Hence (1), (2) and (3) are correct and (2) is false. Ans. (b)

ASYMPTOTES OF ALGEBRAIC CURVES

An asymptote which is not parallel to y-axis is called an **oblique asymptote.**

Let y = mx + c be an asymptote curve of y = f (x), then

$$m = \lim_{\substack{x\to\infty \\ \text{or } y\to\infty}} \frac{y}{x} \text{ and } c = \lim_{\substack{x\to\infty \\ \text{or } y\to\infty}} (y - mx)$$

Working Rule

Suppose y = mx + c is an asymptote of the curve.

Put y = mx + c in the equation of the curve and arrange it in descending of two highest degree terms.

Solve these two equation find m and c.

Put them in the equation y = mx + c to get asymptotes.

2. The asymptotes of $x^3 + 2x^2 y - xy^2 - 2y^3 + 2xy + y - 1 = 0$ are given by

(a) x – y + 1 = 0, x + y – 1 = 0, x + 2y = 0 (b) x – y – 1 = 0, x + y + 1 = 0, x + 2y = 0

(c) x – y + 2 = 0, x + y – 4 = 0, x + 2y = 0 (d) none of these

Sol. Put y = mx + c in the equation of the curve, we get

$x^3 + 2x^2 (mx + c) - x (mx + c)^2 - 2 (mx + c)^3 + 4 (mx + c)^2 + 2x (mx + c) + (mx + c) - 1 = 0$

or, $x^3 (1 + 2m - m^2 - 2m^3) + x^2 (2c - 2mc - 6m^2c + 4m^2 + 2m) + \ldots = 0$

Equating to zero the coefficient of two highest degree terms in x, we have

$1 + 2m - m^2 - 2m^3 = 0$ … (1)

and $c (1 - m - 3m^2) + 2m^2 + m = 0$ … (2)

(1) gives m = 1, – 1, – ½ and c = 1, 1, 0

Hence the asymptotes are

y = x + 1, y = – x + 1, y = – 1/2 x **Answer: (a)**

EXPANSION OF FUNCTIONS (INFINITE SERIES)

Some functions of x can be expanded in ascending powers of x in the form of infinite series.

Maclaurin's series and Taylor's series are generally used for the same.

MACLAURIN'S SERIES (OR THEOREM)

Statement: Let f (x) be a function of x which can be expanded in powers of x and let the expansion be differentiable term by term any number of times, then

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^n}{n!} f^n(0) + \ldots$$

EXAMPLES

1. Expand sin x by Maclaurin's series

Sol. Let y = sin x, then $y_n = \sin\left(x + \frac{n\pi}{2}\right)$.

Putting x = 0, we get $(y)_0 = \sin 0 = 0$ and $(y_n)_0 = \sin(n\pi/2)$ … (i)

Putting n = 1, 2, 3, 4, …in (i) we get,

$(y_1)_0 = \sin ½(\pi) = 1$; $(y_2)_0 = \sin \pi = 0$; $(y_3)_0 = \sin 3/2\pi = -1$; $(y_4)_0 = \sin \pi = 0$, etc.

Now by Maclauring's Series, we have

$y = (y)_0 + x (y_1)_0 + x^2/2! (y_2)_0 + x^3/3! (y_3)_0 + \ldots + x^n/n! (y_n)_0 + \ldots$

$\therefore \sin x = 0 + x (1) + x^2/2! (0) + x^3/3! (-1) + \ldots + x^n/n! \sin n\pi/2 + \ldots$

$= x - x^3/3! + x^5/5! + \ldots + x^n/n! (-1)^{(n-1)/2} + \ldots$, where n is an odd number,

Since $\sin ½ (n\pi) = 0$, if n is even

and $(-1)^{(n-1)/2}$, if n is odd.

2. Expand e^{ax} by Maclaurin's Theorem.

Sol. Let $y = e^{ax}$, then $y_n = a^n e^{ax}$.

$\therefore$ Putting x = 0 we get, $(y)_0 = e^0 = 1$; $(y_n)_0 = a^n e^0 = a^n$ … (i)

Putting n = 1, 2, 3,….. in (i), we have

$(y_1)_0 = a$; $(y_2)_0 = a^2$; $(y_3)_0 = a^3$; $(y_4)_0 = a^4$; etc

$\therefore$ By Maclauring's theorem, we get

$e^x = (y)_0 + x (y_1)_0 + x^2/2! (y_2)_0 + x^3/3! (y_3)_0 + x^n/n! (y_n)_0 + \ldots$

$= 1 + xa + x^2/2!\, a^2 + x^3/3!\, a^3 + \ldots + x^n/n!\, a^n + \ldots$

3. Expand $e^{\sin x}$ by Maclaurin's Theorem.

Sol. $y = e^{\sin x}$ Then, $y_1 = e^{\sin x} \cos x = y \cos x$;

$\therefore\ y_2 = y_1 \cos x - y \sin x$;

$y_3 = y_2 \cos x - 2y_1 \sin x - y \cos x$;

$y_4 = y_3 \cos x - 3y_2 \sin x - 3y_1 \cos x + y \sin x$;

$y_5 = y_4 \cos x - 4y_3 \sin x - 6y_2 \cos x + 4y_1 \sin x + y \cos x$, etc.

Putting x = 0, we get,

$(y)_0 = e^{\sin 0} = e^0 = 1$; $(y_1)_0 = (y)_0 = \cos 0 = 1$; $(y_2)_0 = (y_1)_0 \cos 0 - (y)_0 \sin 0 = 1$;

$(y_3)_0 = (y_2)_0 \cos 0 - 2\,(y_1)_0 \sin 0 - (y)_0 \cos 0 = 0$;

$(y_4)_0 = (y_3)_0 \cos 0 - 3\,(y_2)_0 \sin 0 - 3\,(y_1)_0 \cos 0 + (y)_0 \sin 0 = -3$

$(y_5)_0 = (y_4)_0 \cos 0 - 6\,(y_2)_0 \cos 0 + (y)_0 \cos 0 = -3 - 6 + 1 = -8$

$\therefore$ By Maclaurin's theorem we get

$e^{\sin x} = (y)_0 + x\,(y_1)_0 + x^2/2!\,(y_2)_0 + x^3/3!\,(y_3)_0 + x^4/4!\,(y_4)_0 + x^5/5!\,(y_5)_0 + \ldots$

$= 1 + x\,(1) + x^2/2!\,(1) + x^3/3!\,(0) + x^4/4!\,(-3) + x^5/5!\,(-8) + \ldots$

$= 1 + x + \frac{1}{2} x^2 - 1/8\, x^4 - (1/15)\, x^5 + \ldots$

TAYLOR'S SERIES (OR THEOREM)

Statement - Let f (x + h) be a function of h which can expanded in powers of h, and let the expansion be differentiable any number of times with respect to h, then

$$f(x + h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \ldots + \frac{h^n}{n!} f^{(n)}(x) + \ldots$$

Note: If we put x = 0 and h = x in this result, then we get **Maclaurin's Theorem.**

Other forms of Taylor's Theorem

√ Putting x = a, we have $f(a + h) = f(a) + h f'(a) + h^2/2!\, f''(a) + \ldots + h^n / n!\, f^{(n)}(a) + \ldots$

√ Putting x = h and h = a, we have $f(a + h) = f(h) + a f'(h) + a^2/2!\, f''(h) + \ldots + a^n/n!\, f^{(n)}(h) + \ldots$

√ Putting h = (x – a) in form (1) above we get $f(x) = (a) + (x - a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots + \frac{(x-a)^n}{n!} f^{(n)}(a) + \ldots$

EXAMPLES

1. Show that $\log(x + h) = \log h + x/h - x^2/2h^2 + x^3/3h^3 - \ldots\ldots$

Sol. Here we are to expand in powers of x. Thus we are to use the form

$f(x + h) = f(h) + x f'(h) + x^2/2!\, f''(h) + x^3/3!\, f'''(h) + \ldots$ (i)

Here $f(x + h) = \log(x + h)$

$\therefore f(h) = \log h;\ f'(h) = 1/h;\ f''(h) = -1/h^2;\ f'''(h) = 2/h^3$ etc.

$\therefore$Substituting these values in (i) we get

$$\log(x + h) = \log h + x\left(\frac{1}{h}\right) + \frac{x^2}{2!}\left(-\frac{1}{h^2}\right) + \frac{x^3}{3!}\left(\frac{2}{h^3}\right) + \ldots = \log h + x/h - x^2/2h^2 + x^3/3h^3 + \ldots$$

Hence proved

2. Expand e^x in powers of $(x - 1)$

Sol. Here $f(x) = e^x = e^{(x-1)+1}$.

Here we are to use the following form of Taylor's Theorem

$f(x) = f(a) + (x - a) f'(a) + (x - a)^2/2!\, f''(a) + (x - a)^3/3!\, f'''(a) + \ldots$ (i)

Here $f(x) = e^x$

$\therefore f'(x) = e^x;\ f''(x) = e^x;\ f'''(x) = e^x$; etc.

Putting x = 1, we get $f(1) = e;\ f'(1) = e,\ f''(1) = e,\ f'''(1) = e$ etc.

$\therefore$ From (i), putting a = 1; we get

$f(x) = f(1) + (x - 1) f'(1) + (x - 1)^2/2!\, f''(1) + (x - 1)^3/3!\, f'''(1) + \ldots$ or

$e^x = e + (x - 1) e + (x - 1)^2/2!\, e + (x - 1)^3/3!\, e + \ldots,$

$$= e\left\{1 + (x-1) + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \ldots\right\}$$

PARTIAL DIFFERENTIATION

Definition: Let u be a symbol which has a definite value for every pair of values of x and y, then u is called **a function of two independent variables** and y and is written u = f (x, y).

Partial Differential Coefficients: The partial differential coefficients of f (x, y) with respect to x is defined as $\lim_{\delta x \to 0} \frac{f(x+\delta x, y) - f(x, y)}{\delta x}$, provided this limits exists and is written as $\frac{\partial f}{\partial x}$ or f_x or D_x f.

Therefore the partial differential coefficient of f(x, y) with respect to x is the ordinary differential coefficient of f(x, y) when y is regarded as constant.

The partial differential coefficients of $\frac{\partial f}{\partial x}$ with respect to x and y are $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial y \partial x}$ respectively.

The other notations for $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial x \partial y}$, $\frac{\partial^2 f}{\partial y \partial x}$ and $\frac{\partial^2 f}{\partial y^2}$ are f_{xx}, f_{xy}, f_{yx} and f_{yy} respectively

Note:

1. $\frac{\partial^2 f}{\partial y \partial x}$ means $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$ and $\frac{\partial^2 f}{\partial x \partial y}$ means $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$

2. In all ordinary cases $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial f}{\partial x \partial y}$

EXAMPLES

1. Verify that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial u}{\partial y \partial x}$, when u is equal to x sin y + y sin x.

Sol. Given u = x sin y + y sin x … (i)

Differentiating (i) partially with respect to x regarding y as constant, we get

$\frac{\partial u}{\partial x}$ = sin y + y sin x … (ii)

Differentiating (i) partially with respect to y regarding x as constant, we get

$\frac{\partial u}{\partial y}$ = x cos y + sin x … (iii)

Again, differentiating (ii) partially with respect to y regarding x as constant, we get

$\frac{\partial^2 u}{\partial y \partial x}$ = cos y + cos x … (iv)

And differentiating (iii) partially with respect to x regarding y as constant, we get

$\frac{\partial^2 u}{\partial x \partial y}$ = cos y + cos x … (v)

From (iv) and (v) we get, $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

Hence Proved

2. Verify that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$, when u = log $\left\{\frac{x^2 + y^2}{xy}\right\}$

Sol. $u = \log\left\{\frac{x^2 + y^2}{xy}\right\} = \log(x^2 + y^2) - \log x - \log y$ … (i)

Differentiating (i) partially with respect to x, we get

$$\frac{\partial u}{\partial x} = \frac{1}{x^2 + y^2}.2x - \frac{1}{x} = \frac{2x}{x^2 + y^2} - \frac{1}{x} \quad \ldots \text{(ii)}.$$

Differentiating (i) partially with respect to y, we get

$$\frac{\partial u}{\partial y} = \frac{2y}{x^2 + y^2} - \frac{1}{y} \quad \ldots \text{(iii)}$$

Differentiating (ii) partially with respect to y, we get

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{2x}{x^2 + y^2} - \frac{1}{x}\right) 2x \frac{\partial}{\partial y}\left(\frac{1}{x^2 + y^2}\right) = \frac{-2x}{(x^2 + y^2)^2}. 2y = -4xy / (x^2 + y^2)^2. \quad \ldots \text{(iv)}$$

Differentiating (iii) partially with respect to x, we get

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{2y}{x^2 + y^2} - \frac{1}{y}\right] 2y \frac{\partial}{\partial x}\left(\frac{1}{x^2 + y^2}\right) = 2y\left[\frac{-1}{(x^2 + y^2)^2}.2x\right] = -\frac{4xy}{(x^2 + y^2)} \quad \ldots \text{(v)}$$

Therefore, from (iv) and (v) we get, $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$. Hence proved

3. If f = $\tan^{-1}$ (y/x), verify that $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$

Sol. $f = \tan^{-1}(y/x)$ … (i).

Differentiating (i) partially with respect to x, we get

$$\frac{\partial f}{\partial x} = \frac{1}{1 + (y/x)^2}\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2 + y^2} \quad \ldots \text{(ii)}$$

Differentiating (i) partially with respect to y, we get

$$\frac{\partial f}{\partial y} = \frac{1}{1 + (y/x)^2}.\frac{1}{x} = \frac{x}{x^2 + y^2} \quad \ldots \text{(iii)}$$

Differentiating (ii) partially with respect to y, we get

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{-y}{x^2 + y^2}\right) = \frac{(x^2 + y^2)(-1) - (-y)(2y)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}. \quad \ldots \text{(iv)}$$

Differentiating (iii) partially with respect to x, we get

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)} = \frac{y^2 - x^2}{(x^2+y^2)^2}. \quad \ldots (v)$$

Therefore, from (iv) and (v), we get

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}.$$

Hence proved

4. $u = \sin^{-1}\left(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)$, show that $\frac{\partial u}{\partial x} = -\frac{y}{x}\frac{\partial u}{\partial y}$

Sol. Given $u = \sin^{-1}\left(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)$ .....(i)

Differentiating (i) partially with respect to x, we get,

$$\frac{1}{\sqrt{\left[4\sqrt{(xy)}\right]}}\cdot\frac{2\sqrt{y}}{2\sqrt{x}\left(\sqrt{x}+\sqrt{y}\right)} = \frac{\sqrt{y}}{2\sqrt{x}\left(\sqrt{x}+\sqrt{y}\right)(xy)^{1/4}} \quad \ldots (ii)$$

or $\frac{y}{x}\frac{\partial u}{\partial y} = -\frac{y}{x}\left\{\frac{-\sqrt{x}}{2\sqrt{y}\left(\sqrt{x}+\sqrt{y}\right)(xy)^{1/4}}\right\} = \frac{\sqrt{y}}{2\sqrt{x}\left(\sqrt{x}+\sqrt{y}\right)(xy)^{1/4}} = \frac{\partial u}{\partial x}$,

from (ii) or $x\frac{\partial u}{\partial y} + y\frac{\partial u}{\partial y} = 0$.

Hence Proved

5. If $u = \sin^{-1}(x/y) + \tan^{-1}(y/x)$, then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$.

Sol. Given $u = \sin^{-1}(x/y) + \tan^{-1}(y/x)$ … (i)

Therefore $\frac{\partial u}{\partial x} = \frac{1}{\sqrt{\left\{1-(x/y)^2\right\}}}\cdot\frac{1}{y} + \frac{1}{1+(y/x)^2}\left(-\frac{y}{x^2}\right)$ or therefore $\frac{\partial u}{\partial x} = \frac{x}{\sqrt{\left(y^2-x^2\right)}} = \frac{yx}{x^2+y^2}$ … (ii),

and from (i), $\frac{\partial u}{\partial y} = \frac{1}{\sqrt{\left\{1-(x/y)^2\right\}}}\left(-\frac{x}{y^2}\right) + \frac{1}{1+(y/x)^2}\cdot\frac{1}{x}$ or $y\frac{\partial u}{\partial y} = -\frac{x}{\sqrt{\left(y^2-x^2\right)}} + \frac{xy}{x^2+y^2}$ … (iii)

Adding (ii) and (iii), we get

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$ Hence Proved

6. If u = log ($x^2 + y^2 + z^2$), find the value of $\frac{\partial^2 u}{\partial y \partial z}$

Sol. Given u = log ($x^2 + y^2 + z^2$)

Therefore $\frac{\partial u}{\partial z} = \frac{1}{x^2 + y^2 + z^2}.2z = \frac{2z}{(x^2 + y^2 + z^2)}$.

Therefore $\frac{\partial^2 u}{\partial y \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right) = \frac{\partial}{\partial y}\left[\frac{2z}{x^2 + y^2 + z^2}\right] = 2z\,[-(x^2 + y^2 + z^2)^2\, 2y] = \frac{-4yz}{(x^2 + y^2 + z^2)^2}$

HOMOGENEOUS FUNCTIONS

$a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \ldots\ldots + a_ny^n = x^n$ is an expression in x and y in which every term is of degree n. Such a functions is called **homogeneous function** of x and y of degree n. Also,

$$f(x, y) = a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \ldots\ldots + a_ny^n = x^n\left[a_0 + a_1\left(\frac{y}{x}\right) + a_2\left(\frac{y}{x}\right)^2 + \ldots + a_n\left(\frac{y}{x}\right)^n\right] = x^n f(y/x).$$

Hence every homogeneous function in x and y of degree n can by written as x^n f (y/x)

EULER'S THEOREM ON HOMOGENEOUS FUNCTIONS

Statement: If (x, y) be a homogeneous function in x and y of degree n, then,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

In general if f ($x_1, x_2, x_3 \ldots x_n$) of degree n then $x_1\frac{\partial f}{\partial x_1} + x_2\frac{\partial f}{\partial x_3} + x_3\frac{\partial f}{\partial x_3} + \ldots. + x_n\frac{\partial f}{\partial x_n} = nf.$

EXAMPLES

1. Verify Euler's Theorem for y^n sin (y/x).

Sol. Let u = y^n sin (y/x) , then u is a homogeneous function in x and y of degree n. Then

$\frac{\partial u}{\partial x} = y^n \cos\left(\frac{y}{x}\right)x\left(\frac{-y}{x}\right)$ or $x\frac{\partial u}{\partial x} = -\frac{y^{n-1}}{x}\cos\left(\frac{y}{x}\right)$ … (i)

Also $\frac{\partial u}{\partial y} = y^n \cos\left(\frac{y}{x}\right).\frac{1}{x} + ny^{n-1}\sin\left(\frac{y}{x}\right)$ or $y\frac{\partial u}{\partial y} = \frac{y^{n+1}}{x}\cos\left(\frac{y}{x}\right) + ny^n \sin\frac{y}{x}$ … (ii).

Adding (i) and (ii), we get $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = ny^n \sin\frac{y}{x}$ = nu, which verifies Euler's Theorem for the given function.

2. Which of the following is false?

(A) If $u = x^2 + y^2 + z^2$, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 2u$

(B) If $u = x^3 + y^3 + z^3 + 3xyz$, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 3u$

(C) If u = (y/z) + (z/x) + (x/y), then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 2u$

(D) If $u = \frac{x^2}{2y^3} + \frac{y^2z^2}{x^6}$, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial}{\partial z} = -2u$

Sol. (A) $u = x^2 + y^2 + z^2$. Here u is a homogeneous function in three variables x, y and z of degree 2. Therefore by Euler's Theorem, we get $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 2u$. Hence (A) is true.

(B) $u = x^3 + y^3 + z^3$. Here u is a homogeneous function in three variables x, y and z of degree 3. Therefore by Euler's Theorem, we get $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 3u$. Hence (B) is true.

(C) Given that u = (y/z) + (z/x) + (x/y). Here u is a homogeneous function in variables x, y and z of degree zero. Therefore by Euler's theorem, we get $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial}{\partial z} = 0$. $\therefore$ (C) is not correct.

(D) Given that $u = \frac{x^2}{2y^3} + \frac{y^2z^2}{x^6}$. Here u is a function in x, y and z of degree –2.

Therefore by Euler's theorem, we get $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial}{\partial z} = -2u$. $\therefore$ (D) is correct.

Answer: (C)